Statistical-Computational Tradeoffs in Mixed Sparse Linear Regression

Gabriel Arpino, Ramji Venkataramanan

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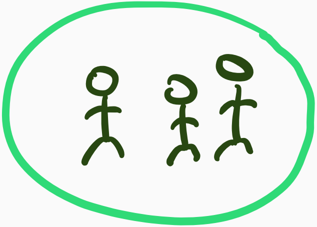
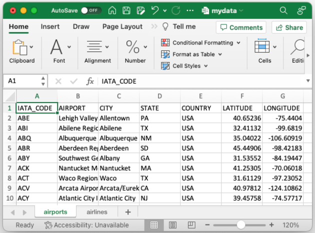
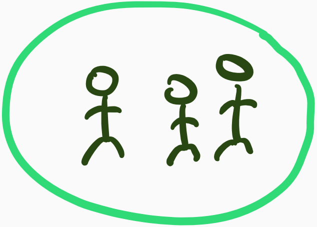
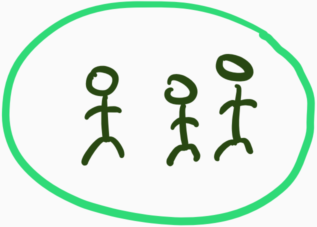
UNIVERSITY OF CAMBRIDGE



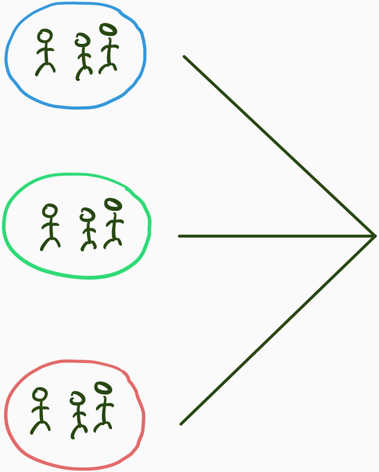


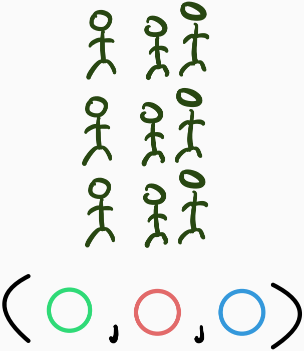
	A	B	C	D	E	F	G
1	IATA_CODE	AIRPORT	CITY	STATE	COUNTRY	LATITUDE	LONGITUDE
2	ABE	Lehigh Valley	Allentown	PA	USA	40.65236	-75.4404
3	ABI	Abilene Regic	Abilene	TX	USA	32.41132	-99.6819
4	ABQ	Albuquerque	Albuquerque	NM	USA	35.04022	-106.60919
5	ABR	Aberdeen Re	Aberdeen	SD	USA	45.44906	-98.42183
6	ABY	Southwest Ge	Albany	GA	USA	31.53552	-84.19447
7	ACK	Nantucket M	Nantucket	MA	USA	41.25305	-70.06018
8	ACT	Waco Region	Waco	TX	USA	31.61129	-97.23052
9	ACV	Arcata Airpor	Arcata/Eurek	CA	USA	40.97812	-124.10862
10	ACY	Atlantic City I	Atlantic City	NJ	USA	39.45758	-74.57717

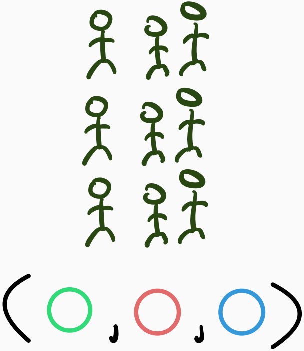




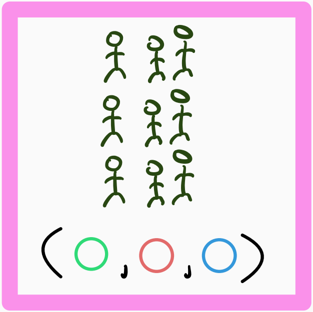
High-Dimensionality

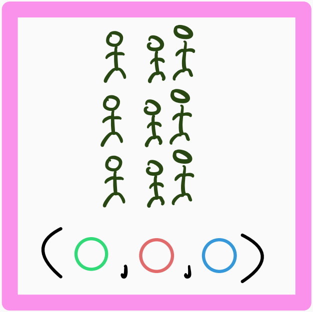




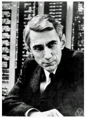
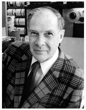


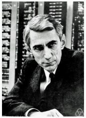
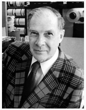
Heterogeneity



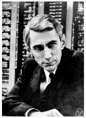
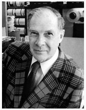


Q: minimal information and computation to recover?

IT Impossible $\mathtt{n}_{\mathtt{IT}}$ Solvable $\mathtt{n}$

$\mathtt{n}$

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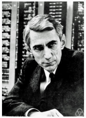
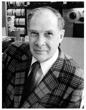
Ex: Communications [S98], Graphical Models [SW09], Regression [MM18], . . .

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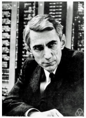
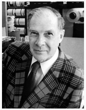

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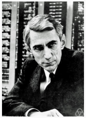
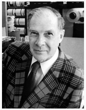
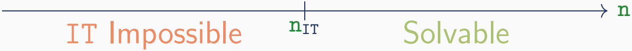
IT Impossible n_{IT} Computational Barrier n_{alg} poly-time n



Ex: Communications [S98], Graphical Models [SW09], Regression [MM18], ...

Ex: Sparse PCA [BR13], Group Testing [CGHWZ22], Sparse Linear Regression [GZ22], ...

Computational Barriers

Failure of efficient families: MCMC, AMP, Stat Query [BMNW22, GJ19, BBHLS21, . . .]

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$$\text{polynomials of degree} \log(\text{dimension}) \text{ w.r.t input} \iff \text{poly-time algorithms}$$

Mixed Sparse Linear Regression (`MSLR`)

Bad News

Good News

Proof

Mixed Linear Regression

$$y_i = \begin{cases} \langle \boldsymbol{x}_i, \boldsymbol{\beta}_1 \rangle + w_i, & \text{with indep. prob. } \phi \\ \langle \boldsymbol{x}_i, \boldsymbol{\beta}_2 \rangle + w_i, & \text{with indep. prob. } 1 - \phi \end{cases}$$

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- $\mathtt{n}$ observations $(y_i, \boldsymbol{x}_i)_{i \in [\mathtt{n}]}$

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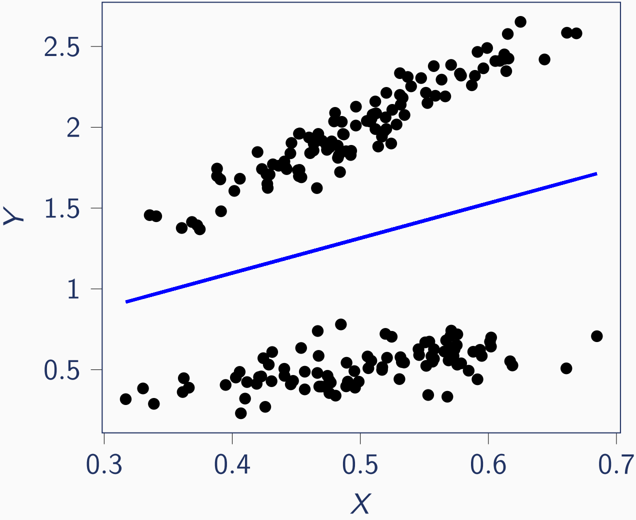
Goal: Recover $\boxed{\boldsymbol{\beta}_1}$, $\boxed{\boldsymbol{\beta}_2}$ w.h.p as $p \to \infty$, given *only* $(y_i, \boldsymbol{x}_i)_{i \in [\mathtt{n}]}$



Applications: *Health Care* [IHHM'22], *Music Perception* [VT'02], *Market Segmentation* [WK'00], ...

Model Assumptions

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Best: ℓ_1-penalization + EM needs $\mathtt{n} = \tilde{\Omega}(k^2)$ [SBG'10], [MP'22]

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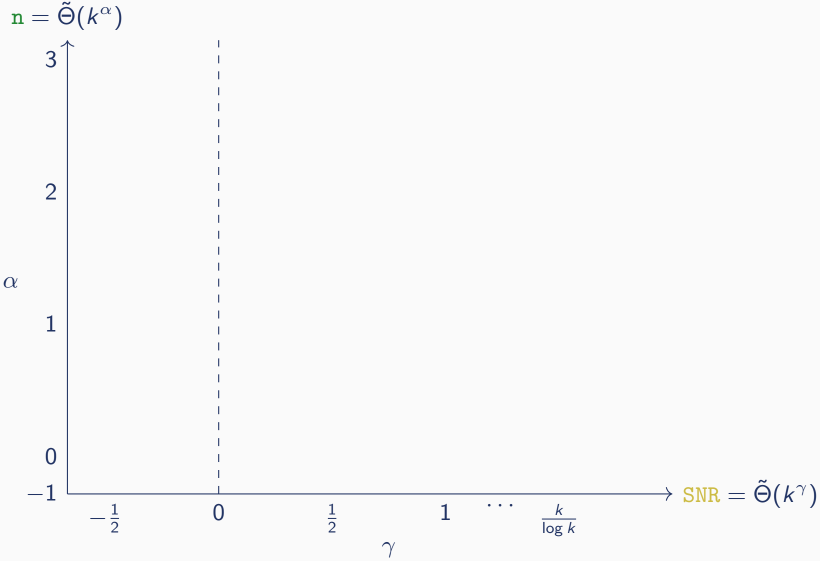
Linear Regression needs just $\mathtt{n} = \tilde{\Omega}(k)$ [RXZ'19], [GZ'22]

Mixed Sparse Linear Regression



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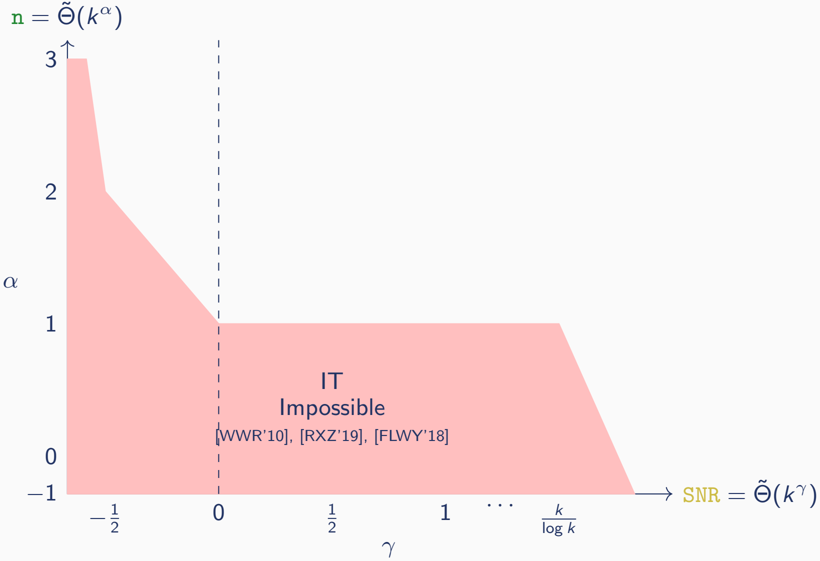

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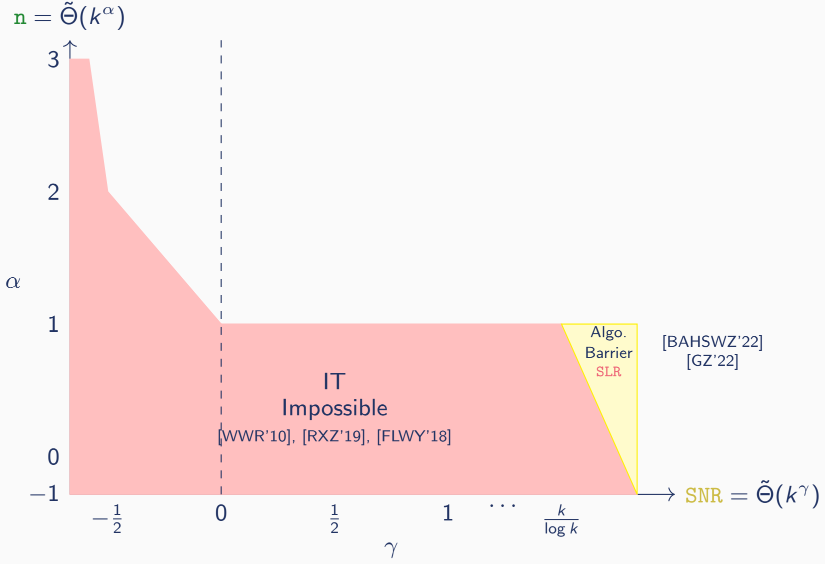

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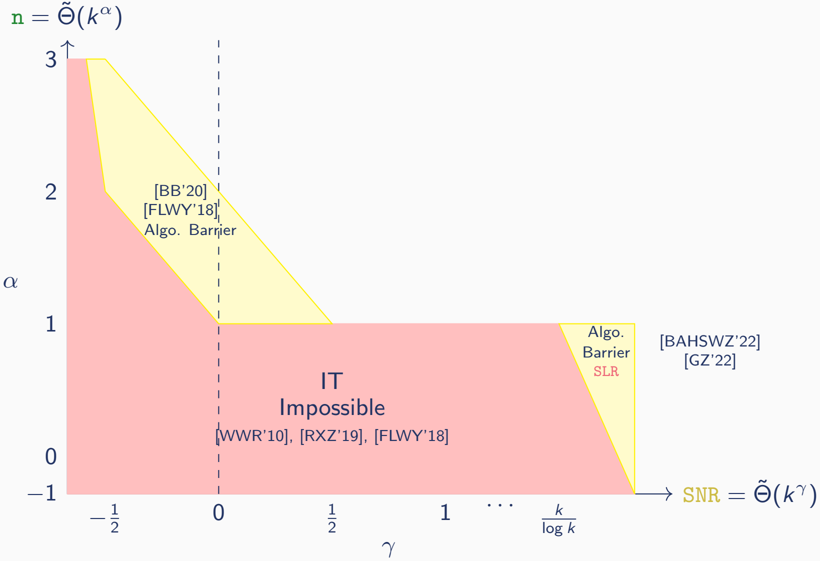

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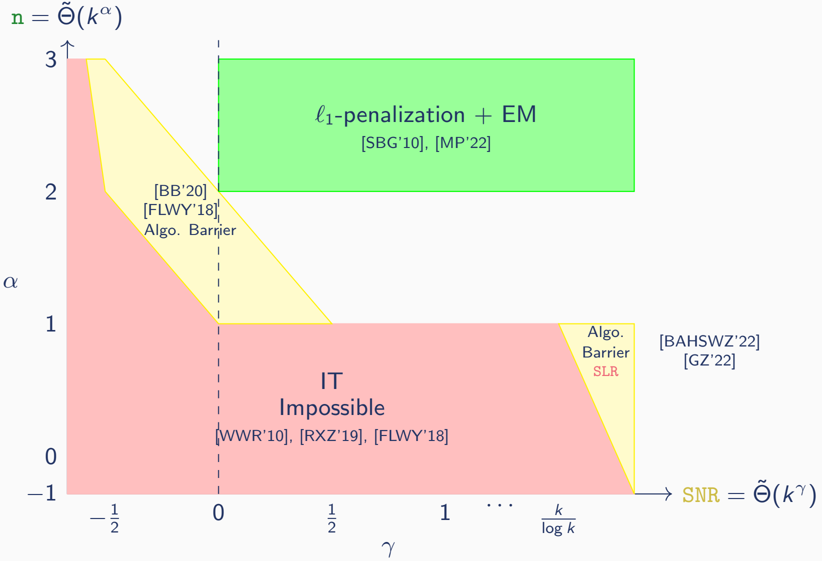

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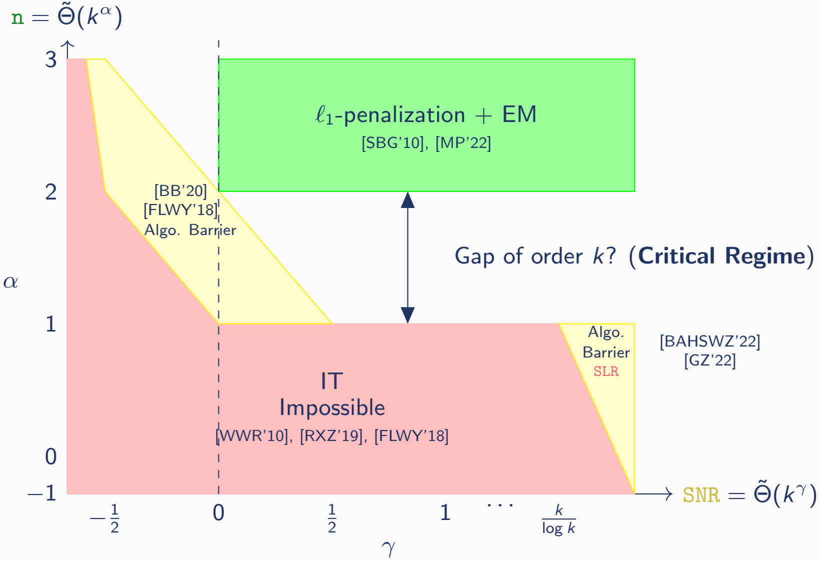

Extreme Cases

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- $\boldsymbol{\beta}_1 = -\boldsymbol{\beta}_2$, $\phi = 1/2$: **Symmetric Mixed Linear Regression**

$$y_i = \begin{cases} \langle \boldsymbol{x}_i, \boldsymbol{\beta} \rangle + w_i, & \text{with prob. } 1/2 \\ -\langle \boldsymbol{x}_i, \boldsymbol{\beta} \rangle + w_i, & \text{with prob. } 1/2 \end{cases}$$

Main Results: Bad News

Theorem. For $k = o(\sqrt{p})$ and $k \ll \mathtt{n} \ll k^2$, analytic polynomials of degree less than $\frac{k^2}{\mathtt{n}}$ cannot solve a hypothesis testing variant of Symmetric Mixed Linear Regression.

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Corollary. For $k = o(\sqrt{p})$ and $k \ll \mathtt{n} \ll k^2$, any randomized algorithm requires runtime $\exp\left(\tilde{\Omega}\left(\frac{k^2}{\mathtt{n}}\right)\right)$ to exactly recover $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ in Symmetric Mixed Linear Regression.

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Low-Degree Conjecture [H18] $+$ Reduction to Recovery

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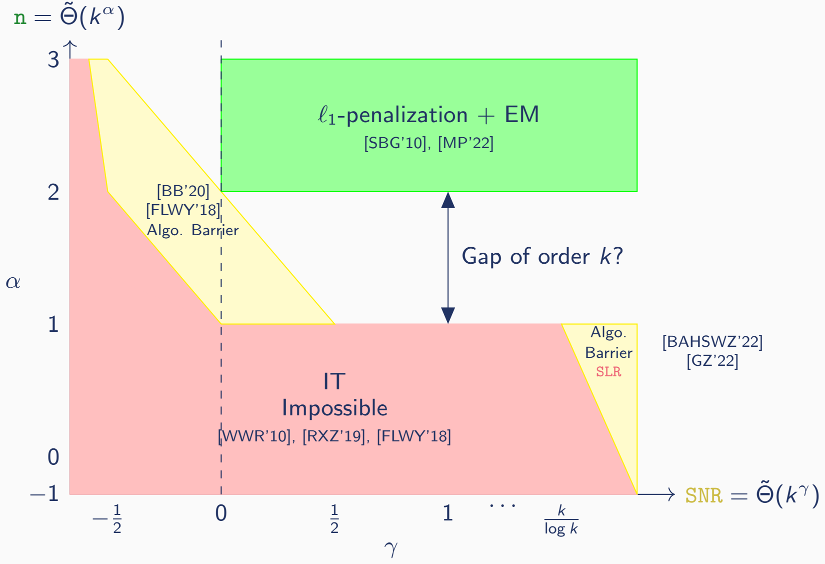
Corollary. For $k = o(\sqrt{p})$ and $k \ll \mathtt{n} \ll k^2$, any randomized algorithm requires runtime $\exp\left(\tilde{\Omega}\left(\frac{k^2}{\mathtt{n}}\right)\right)$ to exactly recover $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ in Symmetric Mixed Linear Regression.

This runtime is *super-polynomial* in the input!

Mixed Sparse Linear Regression



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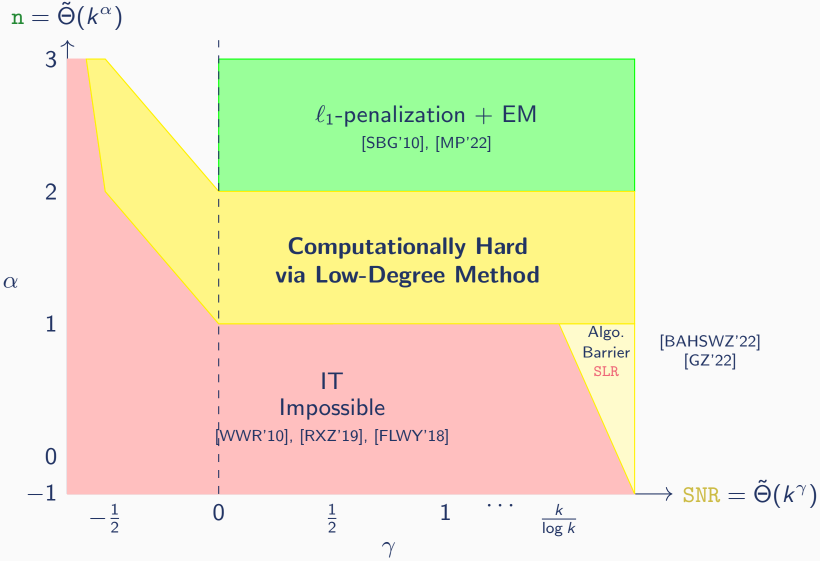

Smooth Tradeoff

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Recover k-sparse $\boldsymbol{x}$ from $y_i \overset{i.i.d}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_p + \beta \boldsymbol{x}\boldsymbol{x}^T)$, $\quad \frac{\mathtt{n}}{p} = \Theta(1)$

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Implications

Symmetric Mixed Linear Regression

$$y_i = \begin{cases} \langle \boldsymbol{x}_i, \boldsymbol{\beta} \rangle + w_i, & \text{with prob. } 1/2 \\ -\langle \boldsymbol{x}_i, \boldsymbol{\beta} \rangle + w_i, & \text{with prob. } 1/2 \end{cases}$$

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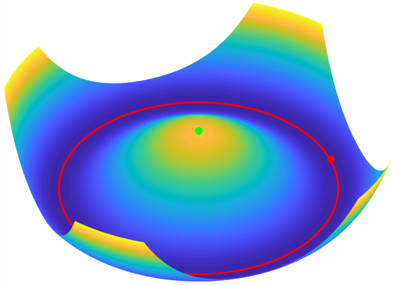

Sparse Phase Retrieval

$$y_i = |\langle \boldsymbol{x}_i, \boldsymbol{\beta} \rangle| + w_i$$

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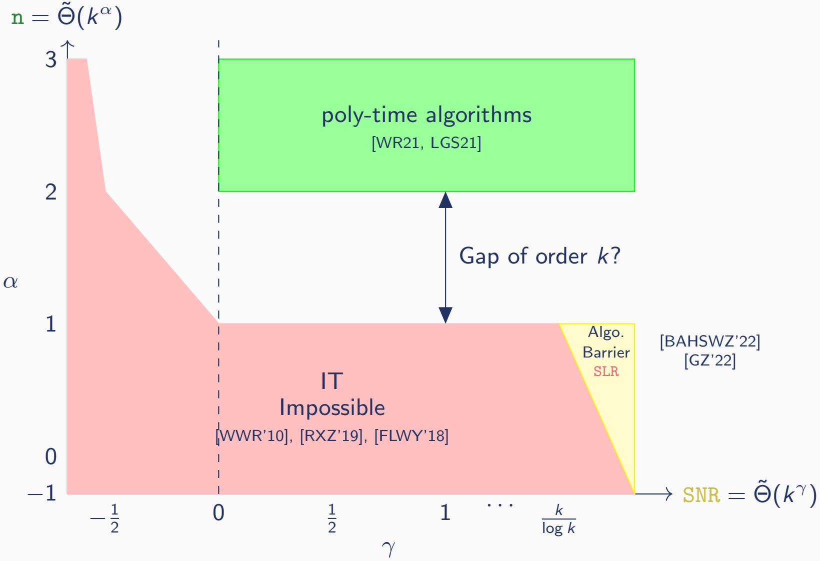

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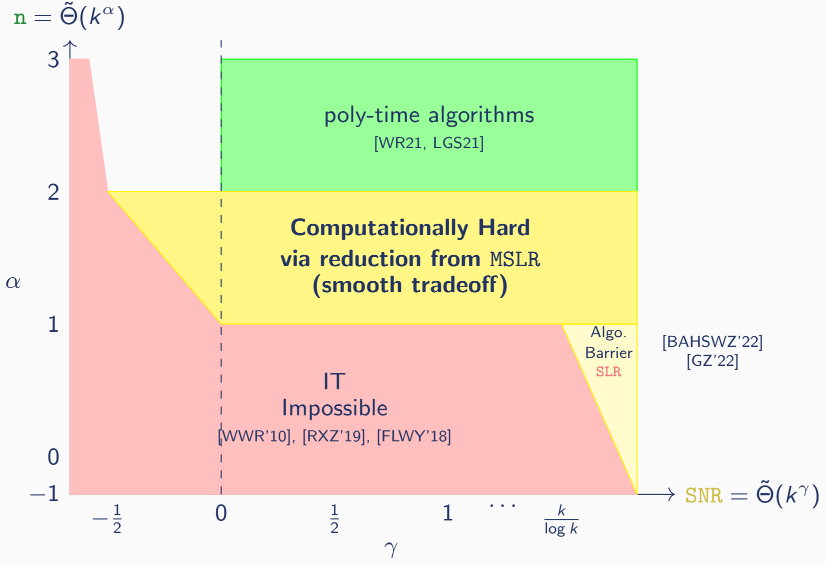

Main Results: Good News

Non-Symmetric Mixed Linear Regression

$$\boldsymbol{\beta}_1 \neq -\boldsymbol{\beta}_2 \text{ or } \phi \neq \frac{1}{2}$$

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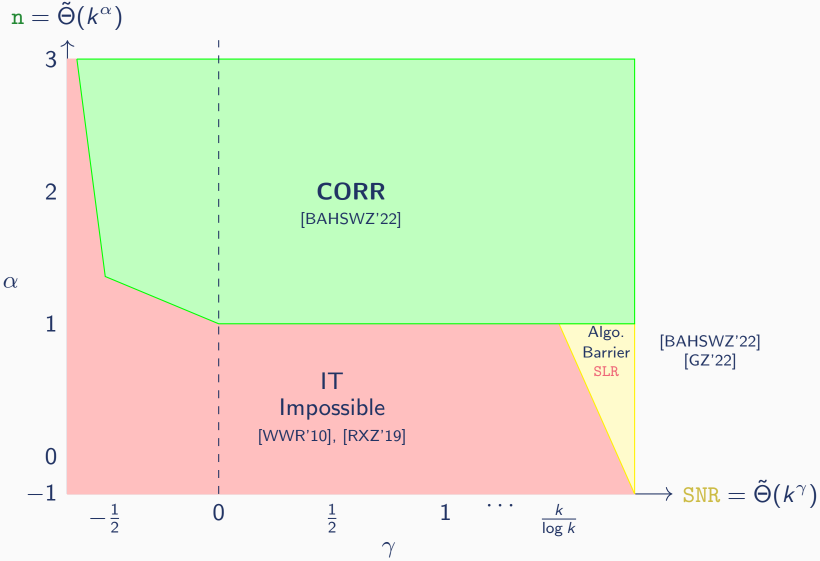

More Good News

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Theorem. Provided $\mathtt{n} = \tilde{\Omega}(k)$, the **CORR** algorithm recovers the joint support of $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ w.h.p.

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Translates to full recovery when combined with dense algorithms [YCS14, CYC14]

Proof Idea

The Low-Degree Method [HS17]

Hypothesis Test between two distributions with $o(1)$ error:

- Null model (pure noise) $\boldsymbol{z} \sim \mathbb{Q}_p$
- Planted model (with "signal") $\boldsymbol{z} \sim \mathbb{P}_p$

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Low-Deg. Conjecture. [H18, CGHWZ22] If $\|L^{\leq D}\| = O(1)$ for some $D = \omega(\log p)$, then any algorithm distinguishing $\mathbb{P}$ from $\mathbb{Q}$ with $o(1)$ error requires runtime $\exp(\tilde{\Omega}(D))$.

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Proposition. [KWB19] The unique solution f^* to the optimization problem

$$\max_{f \text{ deg } D} \frac{\mathbb{E}_{\boldsymbol{z}\sim\mathbb{P}} f(\boldsymbol{z})}{\mathbb{E}_{\boldsymbol{z}\sim\mathbb{Q}}[f(\boldsymbol{z})^2]}$$

is the normalized LDLR $f^* = L^{\le D}/\|L^{\le D}\|$. The value is $\|L^{\le D}\|$.

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Detection Variant

$$y_i = \begin{cases} \langle \boldsymbol{x}_i, \boldsymbol{\beta}_1 \rangle + w_i, & \text{with indep. prob. } \phi \\ \langle \boldsymbol{x}_i, \boldsymbol{\beta}_2 \rangle + w_i, & \text{with indep. prob. } 1 - \phi \end{cases}$$

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Goal: Test whether $(y_i, \boldsymbol{x}_i)_{i \in [\mathtt{n}]}$ came from $\mathbb{P}$ or from $\mathbb{Q}$, with $o(1)$ error as $p \to \infty$.

Proof Notation

$$\boldsymbol{y} = \boldsymbol{X}\beta_1 \odot \boldsymbol{z} + \boldsymbol{X}\beta_2 \odot (1 - \boldsymbol{z}) + \boldsymbol{w}, \quad z_i \overset{i.i.d}{\sim} \text{Bernoulli}(\phi)$$

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$$\boldsymbol{X}_j : \text{column}, \quad \boldsymbol{x}_i : \text{row}$$

$$\boldsymbol{\alpha} \in \mathbb{R}^{n \times (p+1)} : |\boldsymbol{\alpha}| = \sum_{i,j} \alpha_{i,j}, \boldsymbol{\alpha}! = \prod_{i,j} \alpha_{i,j}!$$

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$$\frac{1}{\sqrt{\boldsymbol{\alpha}!}} H_{\boldsymbol{\alpha}}(\boldsymbol{u}) = \frac{1}{\sqrt{\boldsymbol{\alpha}!}} \prod_{i=1}^{\mathtt{n}} \prod_{j=1}^{p+1} H_{\alpha_{i,j}}(u_{i,j}), \quad \boldsymbol{\alpha}, \boldsymbol{u} \in \mathbb{R}^{\mathtt{n} \times (p+1)}$$

$\|L^{\leq D}\|^2$

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&\overset{\text{symm.}}{\approx} \sum_{d}^{D} \mathbb{E}_{\mathbb{P}}[\langle\boldsymbol{\beta}_1^{(1)},\boldsymbol{\beta}_1^{(2)}\rangle^{2d}] = O(1) \text{ for } D\le\frac{k^2}{\mathtt{n}}.
\end{aligned}
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Proof Idea: Good News

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$$\textbf{CORR}(\boldsymbol{X}, \boldsymbol{y}) := \left\{ j \in [p] : \left| \frac{\langle \boldsymbol{X}_j, \boldsymbol{y} \rangle}{\|\boldsymbol{y}\|_2} \right| \geq \sqrt{2(1 + \epsilon/2) \log 2p} \right\}$$

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Summary

- Narrow symmetric case: $\mathtt{n} = \tilde{\Omega}(k^2)$
- Broad non-symmetric case: $\mathtt{n} = \tilde{\Omega}(k)$

Future

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- Sub-exp algorithms for `MSLR`, Sparse Phase Retrieval

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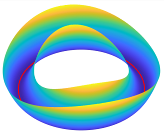
Eigenspace Computation

Compute the principal subspace of a symmetric matrix.



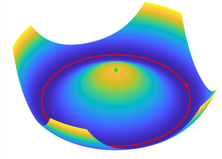
$\min_{\boldsymbol{X}^*\boldsymbol{X}=\boldsymbol{I}} -\frac{1}{2}\mathrm{trace}\left[\boldsymbol{X}^*\boldsymbol{A}\boldsymbol{X}\right].$

Symmetry: $\boldsymbol{X} \mapsto \boldsymbol{X}\boldsymbol{R}$

$\mathbb{G} = \mathsf{O}(r)$

Generalized Phase Retrieval

Recover a complex vector $\boldsymbol{x}_o$ *from magnitude measurements* $\boldsymbol{y} = |\boldsymbol{A}\boldsymbol{x}_o|$.



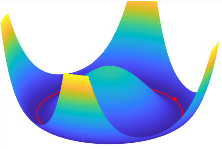
$\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{y}^2 - |\boldsymbol{A}\boldsymbol{x}|^2\|_2^2.$

Symmetry: $\boldsymbol{x} \mapsto \boldsymbol{x}e^{\mathrm{i}\phi}$

$\mathbb{G} = \mathbb{S}^1 \cong \mathsf{O}(2)$

Matrix Recovery

Recover a low-rank matrix $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^*$ *from incomplete/corrupted observations*



$\min_{\boldsymbol{U},\boldsymbol{V}} \mathcal{L}(\boldsymbol{Y} - \mathcal{A}[\boldsymbol{U}\boldsymbol{V}^*]) \ + \ \rho(\boldsymbol{U},\boldsymbol{V}).$

Symmetry: $(\boldsymbol{U},\boldsymbol{V}) \mapsto (\boldsymbol{U}\boldsymbol{\Gamma}, \boldsymbol{V}\boldsymbol{\Gamma}^{-*})$

$\mathbb{G} = \mathsf{GL}(r)$ or $\mathbb{G} = \mathsf{O}(r)$

[WM22]

Appendix: The Low-Degree Method [HS17]

Goal: *Hypothesis Test* between two distributions with $o(1)$ error:

- Null model (pure noise) $\boldsymbol{z} \sim \mathbb{Q}_p$
- Planted model (with "signal") $\boldsymbol{z} \sim \mathbb{P}_p$

Compute

$$\max_{f \deg D} \frac{\mathbb{E}_{\boldsymbol{z}\sim\mathbb{P}}[f(\boldsymbol{z})]}{\sqrt{\mathbb{E}_{\boldsymbol{z}\sim\mathbb{Q}}[f(\boldsymbol{z})^2]}} \qquad \frac{\text{mean in } \mathbb{P}}{\text{fluctuations in } \mathbb{Q}}$$

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- Null model (pure noise) $\boldsymbol{z} \sim \mathbb{Q}_p$
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Can we find a low-degree polynomial $f(\boldsymbol{z})$ that is big for $\boldsymbol{z} \sim \mathbb{P}$ and small for $\boldsymbol{z} \sim \mathbb{Q}$?

Compute

$$\max_{f \deg D} \frac{\mathbb{E}_{\boldsymbol{z}\sim\mathbb{P}}[f(\boldsymbol{z})]}{\sqrt{\mathbb{E}_{\boldsymbol{z}\sim\mathbb{Q}}[f(\boldsymbol{z})^2]}} \qquad \frac{\text{mean in } \mathbb{P}}{\text{fluctuations in } \mathbb{Q}}$$

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$$\max_{f \deg D} \frac{\mathbb{E}_{\boldsymbol{z} \sim \mathbb{P}}[f(\boldsymbol{z})]}{\sqrt{\mathbb{E}_{\boldsymbol{z} \sim \mathbb{Q}}[f(\boldsymbol{z})^2]}} = \max_{f \deg D} \frac{\mathbb{E}_{\boldsymbol{z} \sim \mathbb{Q}}[Lf(\boldsymbol{z})]}{\sqrt{\mathbb{E}_{\boldsymbol{z} \sim \mathbb{Q}}[f(\boldsymbol{z})^2]}}$$

$$L = \frac{d\mathbb{P}}{d\mathbb{Q}}$$

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$$= \max_{f \deg D} \frac{\langle L, f\rangle}{\|f\|}$$

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Appendix: The Low-Degree Method [HS17]

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Appendix: The Low-Degree Conjecture

Informal. [H18, CGHWZ22] Assume "sufficiently nice" distributions $\mathbb{P}$, $\mathbb{Q}$. If $\|L^{\leq D}\| = O(1)$ for some $D = \omega(\log p)$, then algorithms require runtime $\exp(\tilde{\Omega}(D))$ to distinguish $\mathbb{P}$ from $\mathbb{Q}$.

Symmetric Mixed Linear Detection

$$y_i = \begin{cases} \langle \boldsymbol{x}_i, \boldsymbol{\beta} \rangle + w_i, & \text{with indep. prob. } 1/2 \\ -\langle \boldsymbol{x}_i, \boldsymbol{\beta} \rangle + w_i, & \text{with indep. prob. } 1/2 \end{cases}$$

Under $\mathbb{P}$, observe $\boldsymbol{x}_i \overset{i.i.d}{\sim} \mathcal{N}(\mathbf{0}, \boldsymbol{I}_p)$, and y_i as above.

Under $\mathbb{Q}$, observe $(y_i, \boldsymbol{x}_i) \overset{i.i.d}{\sim} \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} (\|\boldsymbol{\beta}_1\|_2^2 + \sigma^2) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_p \end{bmatrix}\right)$.

Goal: Test whether $(y_i, \boldsymbol{x}_i)_{i \in [n]}$ came from $\mathbb{P}$ or from $\mathbb{Q}$, with vanishing error probability as $p \to \infty$.

More Bad News

Symmetric Mixed Linear Regression: $\mathtt{SNR} = \infty$

$$y_i = \pm\langle \boldsymbol{x}_i, \boldsymbol{\beta}\rangle$$

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$$y_i = \pm\langle \boldsymbol{x}_i, \boldsymbol{\beta}\rangle \overset{|\cdot|}{\Longleftarrow} |y_i| = |\langle \boldsymbol{x}_i, \boldsymbol{\beta}\rangle|$$

Corollary. (Conditional on LDC) For $k = o(\sqrt{p})$ and $k \lesssim \texttt{n} \lesssim k^2$, runtime $\exp\left(\frac{k^2}{\texttt{n}}\right)$ is required to solve a detection variant of Sparse Phase Retrieval.

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$\Longrightarrow$ Provides rigorous evidence for computational hardness of Sparse Phase Retrieval when $n = \tilde{o}(k^2)$!